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DIVISIBILITY AND CO-DIVISIBILITY IN MODULES

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Abstract: An exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called $\mathcal{T}$ −pure ($\mathfrak{F}$- copure) if any torsion (torsion free)$R -$ module is projective (injective) relative to it. Since $\mathcal{T}(\mathfrak{F})$ is closed under factors (sub-modules). In this situation **Walker**'s [21] criterion of Co-purity is also applicable. We know that an $R -$ module M is said to be **divisible** with respect to a torsion theory if it is injective relative to any exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with C torsion. Also, an R −module M is said to be **co-divisible** if M is $\mathfrak{F}$ −copure flat module. We denote the torsion sub-module of $A \subseteq M$ by $\sigma(A)$. Walker proved that the class of $\mathcal{I} -$ pure ($\mathfrak{F} -$ copure) sequences form a proper class whenever $\mathcal{I}(\mathfrak{F})$ is closed under homomorphic images (sub-modules) of an $R -$ module M and if $\mathcal{I}(\mathfrak{F})$ is closed under factors (sub-modules). In this case Walker's $\mathcal{I} -$ purity ($\mathfrak{F} -$ copurity) coincides with the earlier notion of purity. We also study about class of R −modules dual to the modules of $\mathfrak{B}$. A sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{I} -$ pure ($\mathfrak{F} -$ copure) if and only if given $C' \leq C \in \mathcal{I}$, there exists $B' \leq B$ such that $B' \cong C'$ and $A \cap B' = 0$. We consider an another notion of purity stronger than the **Cohn**'s purity [12]. If $\mathcal{FG}$ denotes the class of all finitely generated R −modules, since, this class is closed under factors. We deal with purity and co-purity with respect to classes of modules arising out of a given torsion theory on an R-module M. In examining the notion of purities and copurities determined by torsion modules, cyclic torsion modules and torsionfree modules.We also study about divisible modules and co-divisible modules. we try to specify $\mathcal{T}$ −pure injective and $\mathcal{T}$ −pure projective modules and also we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of divisible and co-divisible $R -$ modules. We try to give some alternative easy proofs of some of the results of **Walker** [21] and **Stenstrom**[18] on $\mathcal{I} -$ purity, $\mathcal{I} -$ pure projectivity, and $\mathcal{I} -$ pure injectivity. Here divisible and codivisible modules occur as absolutely $\mathcal{I}(\mathcal{J}_1) -$ pure and absolutely $\mathcal{J} -$ copure modules. Most of these results of the theorem are proved by **Lambek** [17] for $\mathcal{J}_1 -$ purity. We try to give the inter relationship between divisible modules and co-divisible modules.

Key words: $R-$ modules, torsion modules, $\sigma-$ pure projective $R-$modules, $\sigma-$ pure injective $R-$modules, $\mathcal{I}-$pure ($\mathfrak{F}-$ copure), $\mathcal{FG}-$flat modules, Divisible modules, co-divisible modules

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1. Introduction

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. We say that an $R-$ module M is absolutely pure, (respectively regular, flat) with respect to the purity if any short exact sequence with M as the first (respectively second, third) position is pure in the given sense. Now we take a free presentation of N where N is a right $R-$ module and $\bigoplus_J R \xrightarrow{\mu} \bigoplus_I R \longrightarrow N \longrightarrow 0$. We take all the sub-matrices associated with μ are of the column finite matrix. The class of all co-kernels of the right $R-$ maps between $\bigoplus_J R$ and $\bigoplus_I R$ induced by these sub-matrices is denoted by $\wp(N)$. Now we take all row finite sub-matrices of the matrix and take co-kernels of all left $R-$maps between $\bigoplus_I R$ and $\bigoplus_J R$ induced by these sub-matrices and this class of left $R-$ modules is denoted by $\mathcal{L}(N)$. An exact sequence E is called $\mathcal{T}-$pure ($\mathfrak{F}$- copure) if any torsion (torsion free) module is projective (injective) relative to it. Since $\mathcal{T}(\mathfrak{F})$ is closed under factors (sub-modules). In this situation Walker's criterion of Co-purity is applicable. The notation of an $R-$ module M is $\mathcal{T}-$pure projective ($\mathfrak{F}$- copure injective) if and only if $Pext_{\mathcal{T}}(M,A)=0$ ($Pext_{\mathfrak{F}}(A,M)=0$) for all $A\subseteq M$. In particular $Pext_{\mathcal{T}}(T,A)=0$ for all $T\in\mathcal{T}$. We denote the torsion sub-module of $A\subseteq M$ by $\sigma(A)$. Walker proved that the class of $\mathcal{I}-$pure ($\mathcal{J}-$ copure) sequences form a proper class whenever $\mathcal{I}(\mathcal{J})$ is closed under homomorphic images (sub-modules) of an $R-$ module M and if $\mathcal{I}(\mathcal{J})$ is closed under factors (sub-modles) then for any $\mathcal{I}-$pure ($\mathcal{J}-$ copure) sequence $E: 0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ and $E\in\pi^{-1}(\mathcal{I})$ ($E\in i^{-1}(\mathcal{I})$). Hence, in this case Walker's $\mathcal{I}-$purity ($\mathcal{J}-$ copurity) coincides with the earlier notion of purity. We also study about class of $R-$modules dual to the modules of $\mathfrak{B}$. A sequence $E: 0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ is $\mathcal{I}-$pure ($\mathcal{J}-$ copure) if and only if given $C'\leq C\in\mathcal{I}$, there exists$B'\leq B$ such that $B'\cong C'$ and $A\cap B'=0$. We consider an another notion of purity stronger than the **Cohn**'s purity [12]. If $\mathcal{FG}$ denotes the class of all finitely generated $R-$modules, since, this class is closed under factors. We deal with purity and co-purity with respect to classes of modules arising out of a given torsion theory on an R- module M. In examining the notion of purities and copurities determined by torsion modules, cyclic torsion modules and torsionfree modules.We also study about divisible modules and co-divisible modules. we try to specify $\mathcal{T}-$pure injective and $\mathcal{T}-$pure projective modules and also we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of it divisible and co-divisible $R-$ modules. We try to give some alternative easy proofs of some of the results of **Walker** [21] and **Stenstrom** [18] on $\mathcal{I}-$purity, $\mathcal{I}-$pure projectivity, and $\mathcal{I}-$pure injectivity. Here divisible and co-divisible modules occure as absolutely $\mathcal{I}(\mathcal{J}_1)-$pure and absolutely $\mathcal{J}-$ copure modules. Most of these results of the theorem are proved by **Lambek** [17] for $\mathcal{J}_1-$purity. In this present

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paper we try to give the inter relationship between divisible modules and co-divisible modules.Here we give some definitions which are used or related to this present paper

Definition: **1**. An $R-$ module M is said to be **cyclic** if and only if there exists an element $m_0 \in M$ such that $M = Rm_0$.

2. An $R-$ module M is said to be **finitely generated** if and only if there exists a finite generating set X of M.

3. A left $R-$ module M is said **to finitely co-generated** if and only if for each set $\{U_i \,|i \in I\}$ of submodules U_i of M with $\cap_{i\in I} U_i = 0$, there exists a finite subset $\{U_i \,|i \in I_0\}$ that is $I_0 \subset I$ and I_0 is finite with $\cap_{i\in I} U_i = 0$. In other words we can say A module M is said to be **finitely co-generated** if it is co-generated by the family $\{E(S_{i\in I})\}$ finitely. That is $E(M) = \bigoplus_{i=1}^{n} E(S_i)$ where $S_{i\in I}$ are simple modules not necessarily non- isomorphic.

4. An $R-$ module M is said to be **cocyclic** if it is contained in $E(S)$ for some simple module S, where $E(S)$ is a family of co-generators for each R module M.

5. The pair (ϕ, f) is said to be the **pullback** of the pair (ψ, g) if and only if for every pair (ϕ', f') with $\phi': Y \to M,\ f': Y \to B$ and $(\psi o \phi') = (g o f')$, there exists a unique map $\tau: Y \to A$ such that $(f o \tau) = f' and\ (\phi o \tau) = \phi'$.

6. An $R-$ module M is said to be **finitely presented** if there is an exact sequence $M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$ where M_0 and M_1 are free modules with finite bases.

7. Let R be a ring and M is a left $R-$ module, then M is said to **flat** if for every

exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N" \longrightarrow 0$ and the transformed sequence

$0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N" \longrightarrow 0$ is exact.

8. A ring R is **hereditary** if and only if every ideal is a projective module.

9. A **torsion theory** is a pair $(\mathcal{I}, \mathfrak{F})$ of classes of modules satisfying:

(i). $Hom(T, F) = 0, \forall\, T \in \mathcal{I}$ and $F \in \mathfrak{F}$

(ii). $Hom(L, F) = 0, \forall\, F \in \mathfrak{F} \Rightarrow L \in \mathcal{I}$

(iii). $Hom(T, N) = 0, \forall\, T \in \mathcal{I} \Rightarrow N \in \mathfrak{F}$

The classes $\mathfrak{F}$ and $\mathcal{I}$ are known as **torsion free** and **torsion** classes associated with a torsion theory$(\mathcal{I}, \mathfrak{F})$. A torsion theory $(\mathcal{I}, \mathfrak{F})$ is said to be **hereditary** if and only if $\mathcal{I}$ is closed under homomorphic images, direct sums, extensions and submodules. Similarly, $\mathfrak{F}$ is closed under submodules, direct products, extensions and injective envelipes.

10. A left $R-module$ P is said to be $\boldsymbol{\sigma-}$ **pure projective** module if it is projective to relative to every $\sigma-$ pure epimorphism. That is given any $\sigma-$ pure exact sequence

$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and a homomorphism $f: P \longrightarrow C$, there exists a map $h: P \longrightarrow B$ such that $poh = f$ where $p: B \longrightarrow C$ be an onto homomorphism.

11. A left $R - module$ Q is said to be **finitely $\sigma-$ pure injective** if it is $(\mathcal{FG}, \sigma)$- pure in every pure extension of Q. That is if $0 \longrightarrow Q \longrightarrow Q' \longrightarrow Q" \longrightarrow 0$ is a pure exact sequence then it is $(\mathcal{FG}, \sigma)$- pure also.Similarly, Q is said to be **cyclically $\sigma-$ pure injective** if it is cyclically $\sigma -$ pure in every pure extension of it.

12. A sub-module A of an R-module B is called **closed** if $B|A$ is torsion free and it is called dense if $B|A$ is torsion. Any closed submodule A of an R-module B is $\mathcal{T}$ $-$pure.

13. Given a class of modules $\mathcal{I}(\mathfrak{F})$, a sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called **$\mathcal{I}-$ pure** ($\mathfrak{F}-$ **copure**) if A is a direct summand of D whenever $A \leq D \leq B$ and $D|A \in \mathcal{I}$ (A|S is a direct summand of B|S whenever $S \leq A$ and $A|S \in \mathfrak{F}$).

2. **Divisibility and Co-divisibility**

Given a torsion theory $(\mathcal{T}, \mathfrak{F})$, an exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called $\mathcal{T}$ $-$pure ($\mathfrak{F}$- copure) if any torsion (torsion free) $R -$ module is projective (injective) relative to it. Since $\mathcal{T}$($\mathfrak{F}$) is closed under factors (sub-modules). In this situation **Walker**'s criterion of Co-purity is also applicable. In the notation of an R- module M is $\mathcal{T}$ $-$pure projective ($\mathfrak{F}$- copure injective) if and only if $Pext_{\mathcal{T}}(M, A) = 0$ ($Pext_{\mathfrak{F}}(A, M) = 0$) for all $A \subseteq M$. In particular $Pext_{\mathcal{T}}(T, A) = 0$ for all $T \in \mathcal{T}$. We denote the torsion submodule of A by $\sigma(A)$. Walker proved that the class of $\mathcal{I} -$ pure ($\mathfrak{F}-$ copure) sequences form a proper class whenever $\mathcal{I}(\mathfrak{F})$ is closed under homomorphic images (sub-modules) of an $R -$ module M and if $\mathcal{I}(\mathfrak{F})$ is closed under factors (sub-modules) then for any $\mathcal{I} -$ pure ($\mathfrak{F} -$ copure) sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $E \in \pi^{-1}(\mathcal{I})$ ($E \in i^{-1}(\mathfrak{F})$) and hence in this case Walker's $\mathcal{I} -$ purity ($\mathfrak{F} -$ copurity) coincides with the earlier notion.

Proposition2.1: If $\mathcal{I}(\mathfrak{F})$ is closed under factors (sub-modules) then a sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{I} -$ pure ($\mathfrak{F} -$ copure) if and only if given $C' \leq C \in \mathcal{I}$, there exists$B' \leq B$ such that $B' \cong C'$ and $A \cap B' = 0$ (given $A|A' \in \mathfrak{F}$, there exists $B' \leq B$such that $B' + A = B$ and $A|A' \cong B|B'$).

Remark 2.2(a). In the particular case Walker proved that there are enough pure projective modules (that is given C, there exists $B \in \pi(\pi^{-1}(\mathcal{I}))$ with $\boldsymbol{E}: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $\mathcal{I}$-pure) and gave explicit form of the subgroup $Pext(C, A)$ of $\mathcal{I}$- pure sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$.

(b) In the case of abelian groups **Walker**'s theory of $\mathcal{I} -$ purity ($\mathfrak{F} -$ copurity) applies because the classes of finitely generated and cyclic groups are both closed under factors and the class of co-cyclic groups is closed under sub-modules. In general the class of finitely, or cyclically presented modules are not closed under factors and **Walker**'s theory does not apply to the study of **Cohn**'s[12] purity. It applies of course if the ring is Noetherian.

(c). Given a class of right R-modules $\mathfrak{B}$, we consider the sequences of left modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ such that $0 \longrightarrow L\otimes A \longrightarrow L\otimes B$ is exact for all $L \in \mathfrak{B}$ and we denote this class by $\tau^{-1}(\mathfrak{B})$. Now $\tau^{-1}(\mathfrak{B})$ is also a proper class. In more generally **Stenström**[18] has given the formulation for $\tau^{-1}(\mathfrak{B})$. Given a right $R-$ module N, we can make it as its dual $N^* = Hom_{\mathbb{Z}}(N, \mathbb{Q}|\mathbb{Z})$ into a left $R-$module. Every $R-$module can be considered as an $R-\mathbb{Z}-$bimodule.

Proposition2.3: If $\mathfrak{B}^*$ denotes the class of $R-$modules dual to the $R-$modules of $\mathfrak{B}$, then $\tau^{-1}(\mathfrak{B}) = i^{-1}(\mathfrak{B}^*) = T^{-1}(\pi^{-1}(\mathfrak{B}))$ where $T^{-1}(\mathcal{E})$ is the class of sequences

$E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ such that $T(E) \in \mathcal{E}$.

Proposition2.4: A sub-module A is $\boldsymbol{\mathcal{T}}$ **−pure** in B if and only if given a torsion module C of $B|A$, there exists a submodule B_1 of B such that $B_1 \cong C$ and $A \cap B_1 = 0$.

Note: Proof of this proposition is based on the above proposition.

In particularly, we see that $\tau^{-1}(\mathfrak{B})$ and hence $i^{-1}(\mathfrak{B}^*)$ is closed under direct limits and existence of enough $\pi^{-1}(\mathfrak{B})$- projective modules guarantees of existence of enough $\tau^{-1}(\mathfrak{B})$-injective modules and also, $\tau^{-1}(\mathfrak{B})$- injective modules are direct summand of duals of $\pi^{-1}(\mathfrak{B})$- projective modules (**Stenström**). If we take $\mathfrak{B}$ as the class of finitely presented modules, then this gives the existence of Cohn's pure -injective modules and also specifies all pure injective modules.

Proposition2.5: For any class ϑ, the following statements are equivalent for any $R-$ module M: (i). M is absolutely $\vartheta-$ pure.

(ii). M is injective module relative to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left $R-$ modules with $c \in \vartheta$.
(iii). $Ext(C, M) = 0$ for all $c \in \vartheta$.
(iv). C is $i^{-1}(M)-$ flat for all $c \in \vartheta$.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$

$$\begin{array}{ccccccccc} 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow 0 \end{array}$$

Since, it is given a homomorphism$A \rightarrow M$, we complete the diagram by pushout. Now (ii) is $\vartheta-$ pure and hence homotopy exists, so M is injective relative to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left $R-$ modules with $c \in \vartheta$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$. Given any sequence $0 \longrightarrow M \longrightarrow P \longrightarrow C \longrightarrow 0$, in which M is injective relative to it and hence it splits.

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$.

$$\begin{array}{cccccccc} & 0 \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ \text{E:} & 0 \longrightarrow & M & \longrightarrow & A & \longrightarrow & B & \longrightarrow 0 \end{array}$$

It is given that E and $C \in \vartheta$, then we complete the diagram by pullback. Now by the hypothesis of the upper sequence splits and hence there is a homotopy and hence, the given sequence is $\vartheta -$ pure.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iv})$. It is obvious.

Now dually we have M is $\vartheta -$ copure flat if and only if M is projective with respect to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $A \in \vartheta$ that is if and only if $Ext(M, A) = 0$ for all $A \in \vartheta$. Now we try to specify $\mathcal{T}$ −pure injective and $\mathcal{T}$ −pure projective modules.

Proposition2.6: The following statements are equivalent for any $R -$ module M:
(i). M is $\mathcal{T}$ −pure injective.
(ii). $Pext_{\mathcal{T}}(\mathrm{N}, \mathrm{M}) = 0$ for all R −modules N.
(iii). $Ext(F, M) = 0$ for all $F \in \mathcal{T}_1$.
(iv). M is absolutely $\mathcal{T}_1 -$ pure.
(v). M is injective with respect to closed sub-modules.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Given any $\mathcal{T}$ −pure sequence $0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$, as M is $\mathcal{T}$ −pure injective and so it splits. Hence $Ext(F, M) = 0$ for all $F \in \mathcal{T}_1$.

$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \ldots\ldots\ldots\ldots(1) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \rightleftarrows & P & \rightleftarrows & C & \longrightarrow & 0 \end{array}$$

Suppose that sequence (1) is a $\mathcal{T}$ −pure sequence and $f: A \longrightarrow M$ is given. Now we take a pushout, the lower sequence splits and hence M is $\mathcal{T}$ −pure injective.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$. This statement follows because $Ext(F, M) = Pext_{\mathcal{T}}(F, M)$ for all $F \in \mathcal{T}_1$.

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{ii})$. Conversely, if $Ext(F, M) = 0$ then the sequence $0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$ splits for all $F \in \mathcal{T}_1$.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \sigma(N) & = & \sigma(N) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \ldots\ldots\ldots(1) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \rightleftarrows & P/\sigma(N) & \longrightarrow & N/\sigma(N) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Where, $u: M \longrightarrow P, \pi: P \longrightarrow N, \lambda: N \longrightarrow \frac{N}{\sigma(N)}, \lambda': P \longrightarrow \frac{P}{\sigma(N)}, \mu: M \longrightarrow K,$
$\mu': K \longrightarrow M. i: \sigma(N) \longrightarrow N, j: \sigma(N) \longrightarrow P$ and $q: \frac{P}{\sigma(N)} \longrightarrow K$ be homomorphisms.
It is given that the sequence (1) is $\mathcal{T}$ −pure and we have $j: \sigma(N) \longrightarrow P$ Which is a monomorphism. Now, $N/\sigma(N) \in \mathcal{T}_1$ and hence, the right vertical sequence

$0 \longrightarrow \sigma(N) \longrightarrow N \longrightarrow N/\sigma(N) \longrightarrow 0$ is $\mathcal{T}$ −pure and hence, $\pi' \in \pi^{-1}(\mathcal{T})$ and so the epimorphism π' splits. Now we considering the vertical exact sequence, the identity map above guarantees that the square is a pullback which in turn guarantees that μ is an isomorphism. If $\mu' = \mu^{-1}$, then $\mu' o (q o \lambda') o\, u = (\mu' o q) o (u' o\, \mu) = \mu' o \mu = 1$ and hence, the upper sequence splits also and so, $Pext_{\mathcal{T}}(\mathrm{N}, \mathrm{M}) = 0$ for all R −modules N.

$(\boldsymbol{iii}) \Leftrightarrow (\boldsymbol{iv}) \Leftrightarrow (\boldsymbol{v})$ follows from the previous proposition by taking $\vartheta = \mathcal{T}_1$.

Note: $\mathcal{T}_1$ − purity arises in the theory of torsion free covers (**Teply and Golan** [19]).

Proposition2.7: If for any module $M, M/\sigma(M)$ is projective, then M is $\mathcal{T}$ −pure projective. Conversely, for every $\mathcal{T}$ −pure projective module M, $M/\sigma(M)$ is a projective module provided every torsion free module is a factor of a projective torsion free module.
We know that an R − module M is said to be **divisible** with respect to a torsion theory if it is injective relative to any exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with C torsion.

J. Lambek(17) have given the following proposition which is well known:

Proposition2.8: The following statements are equivalent for any R − module M:

(i). M is divisible.

(ii). M is absolutely $\mathcal{I}$- pure.

(iii). $Ext(T, M) = 0$ for all $T \in \mathcal{I}$.

(iv). $M$is injective with respect to all dense ideals.

(v). M is absolutely $\mathcal{J}_1$ − pure.

(vi). $Ext(T, M) = 0$ for all $T \in \mathcal{J}_1$.

Definition: An R −module M is said to be **codivisible** if M is $\mathfrak{F}$ −copure flat module. The propositions are readily dualization of the above proposition except for the conditions (iv), (v) and (vi)of the above proposition.

Now we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of it divisible and co-divisible R − modules. Most of these results of the theorem are prove d by Lambek [17] for $\mathcal{J}_1$ − purity.

Theorem2.9: Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$……..(1) be an exact sequence. Then

(i). If $N \in \mathfrak{F}$ then the exact sequence (1) is $\mathcal{I}$- pure.

(ii). If $M \in \mathfrak{F}$ and the exact sequence (1) is $\mathcal{I}$- pure then $N \in \mathfrak{F}$.

(iii). If M is divisible and the exact sequence (1) is $\mathcal{I}$- pure, then L is divisible.

(iv). If $M \in \mathfrak{F}$ and L is divisible then $N \in \mathfrak{F}$.

(v). If M is divisible and $N \in \mathfrak{F}$, then $L$is divisible.

Proof: (i). is trivial.

(ii). It is given that a map $f: T \longrightarrow N, T \in \mathcal{I}$ then there exists a map $\mu: T \longrightarrow M$ and $\mu = 0$ as $M \in \mathfrak{F}$ and $\pi: M \longrightarrow N$ be a map such that

$$T$$

$$\swarrow \ \downarrow$$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

Hence, $f = 0 \ and \ N \in \mathfrak{F}$.

(iii). Given that there is an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$....(2)

Also, $C \in \mathcal{I}$, we extend the diagram by divisibility of M.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

Now we see that there exists a homotopy as (1) is $\mathcal{I}$- pure.

(iv). It follows from (ii) also, (1) is $\mathcal{I}$- pure by divisibility of L.

(v). It follows from (iii) also, (1) is $\mathcal{I}$- pure, because $N \in \mathfrak{F}$.

Theorem2.10: Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$........(1) be an exact sequence. Then

(i). If $L \in \mathcal{I}$ then the exact sequence (1) is $\mathfrak{F}$- co-pure.

(ii). If $M \in \mathcal{I}$ and the exact sequence (1) is $\mathfrak{F}$- co-pure then $L \in \mathcal{I}$.

(iii). If M is co-divisible and the exact sequence (1) is $\mathfrak{F}$- co-pure, then N is co-divisible.

(iv). If $M \in \mathcal{I}$ and N is co-divisible then $L \in \mathcal{I}$.

(v). If M is co-divisible and $L \in \mathcal{I}$, then N is co-divisible

Note: Proof of this theorem is same as the proof of the previous theorem in case of

$\mathcal{I}$- purity.

Proposition211: If L is divisible and essential in M then N is torsion free.

Proof:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & \sigma(N) \longrightarrow 0 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

We complete the diagram by pullback. Now homotopy exists by divisibility of L, and hence the upper sequence splits.

Now, here L is essential in M tis implies that L is essential in P and hence $L = P$ and therefore, $\sigma(N) = 0$.

Proposition2.12: If N is co-divisible and L is small in M then L is torsion.

Theorem2.13: A torsion theory $(\mathcal{J}, \mathfrak{F})$ is exact if and only if every torsion free $R -$ module $M$is divisible.

Proof: Suppose that each torsion free module is divisible. Let $T' \subseteq T$ and $T \in \mathcal{J}$, and let $F \in \mathfrak{F}$. Since, F is divisible, then any map $f: T' \longrightarrow F$ extends to a map $g: T \longrightarrow F$ and hence, $f = 0$ as $g = 0$.

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T/T' \longrightarrow 0$$

$$\downarrow \quad \swarrow$$

$$F$$

Hence, $T' \in \mathcal{J}$ and $(\mathcal{J}, \mathfrak{F})$ is a hereditary torsion theory.

Now, let $C \in \mathfrak{F}$ and consider a factor $C"$ of C. We take a map $f: T \longrightarrow C"$ with $T \in \mathcal{J}$.

Now $C \in \mathfrak{F} \Rightarrow C' \in \mathfrak{F}$ and hence C' is divisible and so the exact sequence

$0 \longrightarrow C' \longrightarrow C \longrightarrow C" \longrightarrow 0$ is $\mathcal{J}$- pure. Also,

$$T$$

$$\swarrow \quad \downarrow$$

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C" \longrightarrow 0$$

Where there is a map $g: T \longrightarrow C$ which is the lifting of the map $f: T \longrightarrow C"$ and so, $g = 0$ as $C \in \mathfrak{F}$. Therefore, $f = 0$ and $C" \in \mathfrak{F}$. Hence, $(\mathcal{J}, \mathfrak{F})$ is a co-hereditary torsion theory also.

Conversely, suppose that the torsion theory $(\mathcal{J}, \mathfrak{F})$ is exact, then $\mathfrak{F}$ is closed under factors and injective hulls. Given $M \in \mathfrak{F}$ and $f: A \longrightarrow M$ be an $R -$ homomorphism, where A is dense in B, we extend the diagram by injective hull of M.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \longrightarrow \quad M \longrightarrow E(M) \longrightarrow \quad \frac{E(M)}{M} \longrightarrow 0$$

Where $f: A \longrightarrow M$; $\mu: B \longrightarrow M$; $\lambda: B \longrightarrow C$; $j: M \longrightarrow E(M)$; and $g: B \longrightarrow E(M)$;

$h: C \longrightarrow \frac{E(M)}{M}$; $\pi: E(M) \longrightarrow \frac{E(M)}{M}$ are $R-$ homomorphisms. By hypothesis $E(M) \in \mathfrak{F}$ and $\frac{E(M)}{M} \in \mathfrak{F}$. So $h = 0$ and hence, $\pi o g = 0$. Thus there is a homomorphism $\mu: B \longrightarrow M$ such that $jo\mu = g$. But then $\mu oi = f$; $i: A \longrightarrow B$ and hence M is divisible.

Theorem2.14: If a torsion module M is co-divisible if and only if every torsion module$M$ having a projective cover in an exact torsion theory$(\mathcal{I}, \mathfrak{F})$ is co-divisible.

Proof: This follows dually of the proof of the above theorem.

Conclusion: In examining the notion of purities and copurities determined by torsion modules, cyclic torsion modules and torsion-free modules. We also study about divisible modules and co-divisible modules. we try to specify $\mathcal{T}-$pure injective and $\mathcal{T}-$pure projective modules and also we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of it divisible and co-divisible $R-$ modules. We try to give some alternative easy proofs of some of the results of **Walker** [21] and **Stenstrom** [18] on $\mathcal{I}-$ purity, $\mathcal{I}-$ pure projectivity, and $\mathcal{I}-$ pure injectivity. Here divisible and codivisible modules occure as absolutely $\mathcal{I}(\mathcal{J}_1)-$ pure and absolutely $\mathcal{J}-$ copure modules. Most of these results of the theorem are proved by **Lambek** [17] for $\mathcal{J}_1-$ purity. In this present paper we try to give the inter relationship between divisible modules and co-divisible modules.

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