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Evaluating the Derivatives of Trigonometric Functions with Maple

Chii-Huei Yu

Department of Management and Information, Nan Jeon Institute of Technology, Tainan City, Taiwan
chiihuei@mail.njtc.edu.tw

Abstract
This paper uses the mathematical software Maple for the auxiliary tool to study the differential problem of two types of trigonometric functions. We can obtain the closed forms of any order derivatives of these two types of trigonometric functions by using finite arithmetic-geometric series, Euler's formula and DeMoivre's formula, and hence greatly reduce the difficulty of calculating their higher order derivative values. In addition, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

Keywords: Derivatives, trigonometric functions, closed forms, finite arithmetic-geometric series, Euler's formula, DeMoivre's formula, Maple.

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, [1]-[7] can be adopted as references.

In calculus and engineering mathematics curricula, finding $f^{(n)}(c)$ (the n-th order derivative value of function $f(x)$ at $x=c$), in general, necessary goes through two procedures: Firstly evaluating $f^{(n)}(x)$ (the n-th order derivative of $f(x)$), and secondly substituting $x=c$ to $f^{(n)}(x)$. When evaluating the higher order derivative values of a function (i.e. n is large), these two procedures will make us face with increasingly complex calculations. Therefore, to obtain the answers through manual calculations is not an easy thing. In this paper, we mainly study the differential problem of the following two types of trigonometric functions

$$f(x)=\frac{(a-b)-a\cos x+[a+(n+1)b]\cos nx-(a+nb)\cos(n+1)x}{2(1-\cos x)} \tag{1}$$

$$g(x)=\frac{a\sin x+[a+(n+1)b]\sin nx-(a+nb)\sin(n+1)x}{2(1-\cos x)} \tag{2}$$

, where n is a positive integer, a,b are real numbers, and x is not the multiple of 2π. We can obtain the closed forms of any order derivatives of these two types of trigonometric functions by using finite arithmetic-geometric series, Euler's formula and DeMoivre's formula ; these are the major results in this study (i.e., Theorems 1, 2), and hence greatly reduce the difficulty of calculating their higher order derivative values. As for the related study of differential problems can refer to [8]-[15]. On the other hand, we provide two functions to determine their any order derivatives and some higher order derivative values practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and some formulas used in this study.

Notation.

Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, a,b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z by $\mathrm{Im}(z)$.

Euler's formula.

$e^{iy}=\cos y+i\sin y$, where y is any real number.

DeMoivre's formula.

$(\cos y+i\sin y)^n=\cos ny+i\sin ny$, where n is any integer, y is any real number.

Before deriving the major results in this study, we need to obtain the formula of finite arithmetic-geometric series.

Lemma A. Suppose z is a complex number and $z\neq 1$, n is a positive integer, and a,b are real numbers. Then the finite arithmetic-geometric series

$$\sum_{k=0}^{n}(a+kb)z^k=\frac{a-(a-b)z-[a+(n+1)b]z^{n+1}+(a+nb)z^{n+2}}{(1-z)^2} \tag{3}$$

Proof. Let $h(z)=\sum_{k=0}^{n}(a+kb)z^k$, then $zh(z)=\sum_{k=0}^{n}(a+kb)z^{k+1}$.

It follows that

$$(1-z)h(z)$$

$$=\sum_{k=0}^{n}(a+kb)z^k-\sum_{k=0}^{n}(a+kb)z^{k+1}$$

$$= a + b \cdot \sum_{k=1}^{n} z^k - (a+nb)z^{n+1}$$

$$= a + b \cdot \frac{z(1-z^n)}{1-z} - (a+nb)z^{n+1}$$

$$= \frac{a-(a-b)z-[a+(n+1)b]z^{n+1}+(a+nb)z^{n+2}}{1-z} \tag{4}$$

Thus,

$$h(z) = \frac{a-(a-b)z-[a+(n+1)b]z^{n+1}+(a+nb)z^{n+2}}{(1-z)^2} \qquad \blacksquare$$

The following is the first result in this study; we obtain the closed forms of any order derivatives of function (1).

Theorem 1. Suppose x is not the multiple of 2π, m,n are positive integers, and a,b are real numbers. Let the domain of $f(x) = \frac{(a-b)-a\cos x+[a+(n+1)b]\cos nx-(a+nb)\cos(n+1)x}{2(1-\cos x)}$ be $\{x \in R | x \neq 2p\pi, p \in Z\}$. Then the m-th order derivative of $f(x)$,

$$f^{(m)}(x) = \sum_{k=0}^{n} k^m (a+kb)\cos\left(kx + \frac{m\pi}{2}\right) \tag{5}$$

Proof. Because

$$f(x) = \frac{(a-b)-a\cos x+[a+(n+1)b]\cos nx-(a+nb)\cos(n+1)x}{2(1-\cos x)}$$

$$= \mathrm{Re}\left(\frac{ae^{-ix}-(a-b)-[a+(n+1)b]e^{inx}+(a+nb)e^{i(n+1)x}}{-2(1-\cos x)}\right)$$

$$= \mathrm{Re}\left(\frac{a-(a-b)e^{ix}-[a+(n+1)b]e^{i(n+1)x}+(a+nb)e^{i(n+2)x}}{1-2e^{ix}+e^{i2x}}\right)$$

$$= \mathrm{Re}\left(\frac{a-(a-b)e^{ix}-[a+(n+1)b](e^{ix})^{n+1}+(a+nb)(e^{ix})^{n+2}}{(1-e^{ix})^2}\right) \quad \text{(by DeMoivre's formula)}$$

$$= \mathrm{Re}\left(\sum_{k=0}^{n}(a+kb)e^{ikx}\right) \quad \text{(Using (3) of Lemma A)}$$

$$= \sum_{k=0}^{n}(a+kb)\cos kx \quad \text{(By Euler's formula)} \tag{6}$$

It follows that any m-th order derivative of $f(x)$,

$$f^{(m)}(x) = \sum_{k=0}^{n} k^m (a+kb)\cos\left(kx + \frac{m\pi}{2}\right) \qquad \blacksquare$$

Next, we determine the closed forms of any order derivatives of function (2).

Theorem 2. If the assumptions are the same as Theorem 1, and suppose the domain of

$g(x)=\dfrac{a\sin x+[a+(n+1)b]\sin nx-(a+nb)\sin(n+1)x}{2(1-\cos x)}$ be $\{x\in R | x\neq 2p\pi, p\in Z\}$. Then the m-th order derivative of $g(x)$,

$$g^{(m)}(x)=\sum_{k=0}^{n}k^m(a+kb)\sin\left(kx+\frac{m\pi}{2}\right) \tag{7}$$

Proof. Because

$$g(x)=\frac{a\sin x+[a+(n+1)b]\sin nx-(a+nb)\sin(n+1)x}{2(1-\cos x)}$$

$$=\operatorname{Im}\left(\frac{ae^{-ix}-(a-b)-[a+(n+1)b]e^{inx}+(a+nb)e^{i(n+1)x}}{-2(1-\cos x)}\right)$$

$$=\operatorname{Im}\left(\frac{a-(a-b)e^{ix}-[a+(n+1)b](e^{ix})^{n+1}+(a+nb)(e^{ix})^{n+2}}{(1-e^{ix})^2}\right)$$

$$=\operatorname{Im}\left(\sum_{k=0}^{n}(a+kb)e^{ikx}\right)\quad\text{(Using (3) of Lemma A)}$$

$$=\sum_{k=0}^{n}(a+kb)\sin kx\quad\text{(By Euler's formula)} \tag{8}$$

It follows that any m-th order derivative of $g(x)$,

$$g^{(m)}(x)=\sum_{k=0}^{n}k^m(a+kb)\sin\left(kx+\frac{m\pi}{2}\right)$$

3. Examples

In the following, we provide two functions to determine the closed forms of their any order derivatives and evaluate some of their higher order derivative values practically. On the other hand, we use Maple to calculate the approximations of these higher order derivative values and their closed forms for verifying our answers.

Example 1. Suppose the domain of the trigonometric function

$$f(x)=\frac{3-5\cos x+13\cos 3x-11\cos 4x}{2(1-\cos x)} \tag{9}$$

is $\{x\in R | x\neq 2p\pi, p\in Z\}$ (the case of $a=5, b=2, n=3$ in Theorem 1).

By Theorem 1, we obtain any m-th order derivative of $f(x)$,

$$f^{(m)}(x)=\sum_{k=0}^{3}k^m(5+2k)\cos\left(kx+\frac{m\pi}{2}\right) \tag{10}$$

for all x is not the multiple of 2π.

Thus, we can determine the 12-th order derivative value of $f(x)$ at $x=\dfrac{5\pi}{6}$,

$$f^{(12)}\left(\frac{5\pi}{6}\right)=\sum_{k=0}^{3}k^{12}(5+2k)\cos\frac{5k\pi}{6} \tag{11}$$

In the following, we use Maple to verify the correctness of (11).

>f:=x->(3-5*cos(x)+13*cos(3*x)-11*cos(4*x))/(2*(1-cos(x)));

$$f := x \to \frac{3 - 5\cos(x) + 13\cos(3x) - 11\cos(4x)}{2 - 2\cos(x)}$$

>evalf((D@@12)(f)(5*Pi/6),26);

18425.93782217350892951

>evalf(sum(k^12*(5+2*k)*cos(5*k*Pi/6),k=0..3),24);

18425.9378221735089294727

Example 2. Assume the domain of the trigonometric function

$$g(x) = \frac{7\sin x - 26\sin 10x + 23\sin 11x}{2(1-\cos x)} \tag{12}$$

is $\{x \in R \mid x \neq 2p\pi, p \in Z\}$ (the case of $a = 7, b = -3, n = 10$ in Theorem 2).

Using Theorem 2, we can determine any m -th order derivative of $g(x)$,

$$g^{(m)}(x) = \sum_{k=0}^{10} k^m (7-3k)\sin\left(kx + \frac{m\pi}{2}\right) \tag{13}$$

for all x is not the multiple of 2π .

Hence, we obtain the 23-th order derivative value of $g(x)$ at $x = -\frac{3\pi}{4}$,

$$g^{(23)}\left(-\frac{3\pi}{4}\right) = -\sum_{k=0}^{10} k^{23}(7-3k)\cos\frac{3k\pi}{4} \tag{14}$$

We also use Maple to verify the correctness of (14).

>g:=x->(7*sin(x)-26*sin(10*x)+23*sin(11*x))/(2*(1-cos(x)));

$$g := x \to \frac{7\sin(x) - 26\sin(10x) + 23\sin(11x)}{2 - 2\cos(x)}$$

>evalf((D@@23)(g)(-3*Pi/4),28);

$$-1.155767138292301449074268178 \cdot 10^{23}$$

>evalf(-sum(k^23*(7-3*k)*cos(3*k*Pi/4),k=0..10),28);

$$-1.155767138292301449074267996 \cdot 10^{23}$$

4. Conclusion

From the above discussion, we know the finite arithmetic-geometric series, the Euler's formula and the DeMoivre's formula play significant roles in the theoretical inferences of this study. In fact, the applications of these three formulas are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications.

On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics

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